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On initial algebras of multiplicative invariants

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Abstract

Consider the ring of *multiplicative invariants*, $\Bbbk[\mathsf{A}]^{\mathcal{G}}$, of the group algebra $\Bbbk[\mathsf{A}]$ of a faithful $\mathcal{G}$-lattice A over base field $\Bbbk$. Among other things, we will determine the cardinality of the set of all *initial algebras*, $\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathsf{A}]^{\mathcal{G}})$, of the ring of multiplicative invariants over all possible admissible orders $\succcurlyeq$ on A.

Keywords: Initial algebra; Initial convex cone; Gröbner region

1. Introduction

Let A be a faithful $\mathcal{G}$-lattice, i.e. A is a free abelian group of finite rank with injective representation $\mathcal{G} \hookrightarrow \mathrm{GL}(\mathsf{A})$. The $\mathcal{G}$ action on A extends to the group algebra $\Bbbk[\mathsf{A}]$ by $\Bbbk$-algebra automorphism. The elements of A are monomials in the group algebra $\Bbbk[\mathsf{A}]$ and the operation on A will be multiplication. Due to this the $\mathcal{G}$-action on $\Bbbk[\mathsf{A}]$ is called *multiplicative* or *monomial* action. The subalgebra,

$$\Bbbk[\mathsf{A}]^{\mathcal{G}} = \left\{ f \in \Bbbk[\mathsf{A}] \,\middle|\, \varphi(f) = f,\ \forall \varphi \in \mathcal{G} \right\}$$

is called an *algebra* (*ring*) *of multiplicative invariants*. For details on the theory of multiplicative invariants, we direct the reader to [7].

For ease of computations, we identify the free abelian group A of rank n with $\mathbb{Z}^n$ and hence the group algebra $\Bbbk[\mathsf{A}]$ with the Laurent polynomial algebra $\Bbbk[\mathbf{x}^{\pm 1}] = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The

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monomials in $\Bbbk[\mathbf{x}^{\pm 1}]$ are $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. As a result $\mathcal{G}$ will be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, acting on the exponents of each monomial by matrix multiplication.

By an admissible *term* (*monomial*) *order* on $\Bbbk[\mathbf{x}^{\pm 1}]$ we mean be a total order $\succcurlyeq$ on $\mathbb{Z}^n$ satisfying $\mathbf{a} \succcurlyeq \mathbf{b} \Rightarrow \mathbf{a} + \mathbf{c} \succcurlyeq \mathbf{b} + \mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{Z}^n$. Let f be a non-zero Laurent polynomial in $\Bbbk[\mathbf{x}^{\pm 1}]$ and $\succcurlyeq$ a monomial order on $\Bbbk[\mathbf{x}^{\pm 1}]$, the *initial* of f with respect to $\succcurlyeq$ is defined to be

$$\mathrm{in}_{\succcurlyeq}(f) := \max\bigl\{\mathbf{a} \in \mathbb{Z}^n \bigm| \mathbf{x}^{\mathbf{a}} \text{ occurs in } f \text{ with non-zero coefficient}\bigr\}.$$

The initial algebra of the ring of multiplicative invariants is the monomial algebra

$$\mathrm{in}_{\succcurlyeq}\bigl(\Bbbk\bigl[\mathbf{x}^{\pm 1}\bigr]^{\mathcal{G}}\bigr) = \Bbbk\bigl[\mathbf{x}^{\mathrm{in}_{\succcurlyeq}(f)} \bigm| f \in \Bbbk\bigl[\mathbf{x}^{\pm 1}\bigr]^{\mathcal{G}} \setminus \{0\}\bigr].$$

Just as in Gröbner basis theory, where one considers initial ideals, many properties of $\Bbbk[\mathbf{x}]^{\mathcal{G}}$ are inherited by its initial algebra. This is in particular possible due to the introduction of SAGBI basis by Robbiano and Sweedler [12] and independently by Kapur and Madlener [5]. The term SAGBI is an acronym for "*Subalgebra Analogue to Gröbner Basis for Ideals*."

Our interest in the subject of this paper is inspired by the work of Kuroda [6], Reichstein [9], and Thiéry and Thomassé [16]. Let us assume for a moment that $\mathcal{G}$ consists only permutation matrices, i.e. $\mathcal{G} \leqslant S_n$, the permutation group on $\{1, \ldots, n\}$. Göbel [3,4, etc.], wrote intensively on permutation invariants on polynomial rings, $\Bbbk[\mathbf{x}] = \Bbbk[x_1, \ldots, x_n]$. For example, in [3, Lemma 5.6] he showed that the ring of invariants $\Bbbk[\mathbf{x}]^{\mathcal{G}}$ has a finite SAGBI basis under the usual lexicographic order if and only if $\mathcal{G} \cong S_{n_1} \times \cdots \times S_{n_k}$ for some partition $n = n_1 + \cdots + n_k$. He further conjectured in [4] that the same should be true for an arbitrary monomial order. Kuroda [6, Theorem 2.2], Reichstein [9, Theorem 1.6] and Thiéry and Thomassé [16, Theorem 1.2] solved the conjecture independently. Kuroda further showed in [6, Theorem 2.3] that, "*For $\mathcal{G} \leqslant S_n$, the cardinality of the set of distinct initial algebras of* $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$ *over all possible monomial orders on* $\Bbbk[\mathbf{x}^{\pm 1}]$ *is finite and equals to the order of $\mathcal{G}$ iff $\mathcal{G} \cong S_{n_1} \times \cdots \times S_{n_k}$. It is uncountable otherwise*."

Going back to Göbel's conjecture, the result of Reichstein [9, Theorem 1.6] generalizes further the conjecture. One of our main results, Theorem 1.1, stated below is a generalization of Kuroda's theorem on the cardinality of distinct initial algebras of the ring of multiplicative invariants to the level of Reichstein's generalization of Göbel's conjecture.

In Section 3 we deploy all the necessary tools on the geometry and topology of certain "*initial convex cones*" and "*Gröbner regions*" associated to the ring of invariants. Our second result, Theorem 1.2, is a characterization of ring of multiplicative invariants in terms of dimension of their Gröbner region. First the following definition of hyperplane reflection.

Definition 1 *(Reflection group).* $\sigma \in \mathrm{GL}_n(\mathbb{Z})$ is called a *reflection* if it fixes a hyperplane on the vector space $V = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and $\sigma^2 = \mathrm{I}$, where I is the identity transformation on V. We call $\mathcal{G}$ a *reflection group* if it is generated by reflections.

Summarizing, we plan to prove the following two results in Section 4:

Theorem 1.1. *Let $\mathcal{G}$ be a finite subgroup of* $\mathrm{GL}_n(\mathbb{Z})$. *The cardinality of the set of distinct initial algebras,* $\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$, *over all monomial orders $\succcurlyeq$ on* $\Bbbk[\mathbf{x}^{\pm 1}]$ *is finite and equal to $|\mathcal{G}|$ if and only if $\mathcal{G}$ is a reflection group. If $\mathcal{G}$ is a non-reflection group then the cardinality becomes c of the continuum.*

Theorem 1.2. *The dimension of any Gröbner region of $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$ is equal to n if and only if $\mathcal{G}$ is a reflection group. On the other hand for each $\ell \in \{1, \dots, n-1\}$ there exists a non-reflection group $\mathcal{G}_\ell$ such that dimension of Gröbner region of $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_\ell}$ is exactly ℓ.*

2. Monomial orders

2.1. Notations and conventions

Bold letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ etc. denotes n-tuple of points in $\mathbb{Z}^n$ or $\mathbb{R}^n$. We will also use the following notations throughout this paper.

$\succcurlyeq$	an admissible term order on $\mathbb{Z}^n$;
$\mathbf{a} \succ \mathbf{b}$	denotes $\mathbf{a} \succcurlyeq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$;
Ω	denote the set of all (admissible) orders on $\mathbb{Z}^n$;
$\succcurlyeq_{\text{lex}}$	denote the lexicographic order defined on $\mathbb{Z}^n$ by $\mathbf{a} \succcurlyeq_{\text{lex}} \mathbf{b}$ if and only if $\mathbf{a} = \mathbf{b}$ or the first non-zero entry of $\mathbf{a} - \mathbf{b}$ is positive;
$\_ \cdot \_$	denote the usual inner (dot) product on $\mathbb{R}^n$.

2.2. Some properties of monomial orders

In [11, Theorem 4] Robbiano gave a characterization of all the admissible orders on $\mathbb{Z}^n$ in terms of certain order monomorphisms from $\mathbb{Z}^n$ to $\mathbb{R}^s$ with lexicographic order on $\mathbb{R}^s$ where $s \leqslant n$. If $n \geqslant 2$ one can easily deduce from Robbiano's theorem that $|\Omega|$ is the continuum c. In this section we will construct an uncountable subset of Ω that will be useful for our purposes in the rest of this paper. First the following standard fact from linear algebra, (a proof is included for lack of reference).

Lemma 2.1. *A* (*finite-dimensional*) *vector space V over a field $\Bbbk$ of uncountable cardinality cannot be expressed as a countable union of proper subspaces.*

Proof. By contradiction assume that $V = \bigcup_{i=1}^{\infty} V_i$ where V_i are proper subspaces of V. It suffices to show that any finite set of vectors in V is contained in some V_i. We show this by induction on $s \in \mathbb{N}$ for any set $\{\mathbf{u}_1, \dots, \mathbf{u}_s\} \subseteq V$. The case $s = 1$ is trivial. Now by induction assumption, for each $k \in \Bbbk$ there exists $j \in \mathbb{N}$ such that $S_k := \{\mathbf{u}_1, \dots, \mathbf{u}_{s-1} + k\mathbf{u}_s\} \subseteq V_j$. But since $\Bbbk$ is uncountable then some V_j contains S_{k_1} and S_{k_2} for $k_1 \neq k_2$ which implies that $\{\mathbf{u}_1, \dots, \mathbf{u}_s\} \subseteq V_j$. Then, in particular, a finite basis of V would be contained in a proper subspace V_i which would give the contradiction $V_i = V$. $\square$

Let $\mathbf{u} \in \mathbb{R}^n$, define an ordering $\succcurlyeq_{\mathbf{u}}$ on $\mathbb{Z}^n$ by a weight vector $\mathbf{u}$ to be

$$\mathbf{a} \succcurlyeq_{\mathbf{u}} \mathbf{b} \quad :\Leftrightarrow \quad \mathbf{u} \cdot \mathbf{a} > \mathbf{u} \cdot \mathbf{b} \vee (\mathbf{u} \cdot \mathbf{a} = \mathbf{u} \cdot \mathbf{b} \wedge \mathbf{a} \succcurlyeq_{\text{lex}} \mathbf{b}). \tag{2.1}$$

It can easily be verified that $\succcurlyeq_{\mathbf{u}}$ is an admissible order on $\mathbb{Z}^n$. Now consider the set $\{\succcurlyeq_{\mathbf{u}} \mid \mathbf{u} \in \mathbb{R}^n\} \subseteq \Omega$. It is clear from the definition that $\succcurlyeq_{\mathbf{u}} = \succcurlyeq_{\lambda\mathbf{u}}$ for all $\lambda \in \mathbb{R}_{>0}$. But if $n \geqslant 2$, and $\mathbf{u}_1$ and $\mathbf{u}_2$ are non-parallel vectors in $\mathbb{R}^n$ then the set $\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{u}_1 > 0 > \mathbf{v} \cdot \mathbf{u}_2\}$ is a non-empty open convex cone in $\mathbb{R}^n$, hence by the density of rational vectors in $\mathbb{R}^n$ the above set contains some $\mathbf{a} \in \mathbb{Z}^n$. It follows that $\mathbf{a} \succcurlyeq_{\mathbf{u}_1} \mathbf{0}$ and $\mathbf{0} \succcurlyeq_{\mathbf{u}_2} \mathbf{a}$, and hence $\succcurlyeq_{\mathbf{u}_1} \neq \succcurlyeq_{\mathbf{u}_2}$. Now since there are an uncountable cardinality of pairwise non-parallel vectors in $\mathbb{R}^n$, (Lemma 2.1), then $\{\succcurlyeq_{\mathbf{u}} \mid \mathbf{u} \in \mathbb{R}^n\}$

is an uncountable subset of Ω. Next, the following technical lemma about extension of monomial orders.

Lemma 2.2 *(Extension of monomial orders). Any $\succcurlyeq \in \Omega$ can be extended to a linear order, also denoted $\succcurlyeq$ on $\mathbb{R}^n$ with the property that when ever $\mathbf{v}_1 \succcurlyeq \mathbf{v}_2$ then $\mathbf{v}_1 + \mathbf{u} \succcurlyeq \mathbf{v}_2 + \mathbf{u}$ and $r\mathbf{v}_1 \succcurlyeq r\mathbf{v}_2$ for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{\geqslant 0}$.*

The extension in Lemma 2.2 is not unique, see [10]. A complete proof that involves Zorn's lemma and other facts is also given in [14, Lemma 2.1].

Remark 1. There is a canonical extension of $\succcurlyeq_{\text{lex}}$ and $\succcurlyeq_{\mathbf{u}}$; $\mathbf{u} \in \mathbb{R}^n$ from $\mathbb{Z}^n$ to $\mathbb{R}^n$ just by defining them exactly the same way as in $\mathbb{Z}^n$.

3. Convex cones associated to $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$

In this section we will construct two convex cones, *initial convex cones* and *Gröbner regions*, for the ring of invariants. Some of the necessary topological and geometric properties of these cones will also be studied.

Remark 2. We will assume through out that $\mathcal{G}$ is **finite**. Such a restriction indeed does not exclude any multiplicative invariant algebra. See [7, §3.3 and Proposition 3.3.1] for details.

3.1. Initial convex cones

A subset C of $\mathbb{R}^n$ is called a *convex cone* if $\sum \lambda_i \mathbf{c}_i \in C$ for all $\mathbf{c}_i \in C$ and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ almost all zero. If C can be generated by finitely many elements, then it is called a *convex polyhedral cone*. The *dimension* of a cone C, denoted $\dim(C)$, is the dimension of the subspace of $\mathbb{R}^n$ spanned by C.

Definition 2 *(Fundamental domain).* Let G be any group acting on a set X, a *fundamental domain* for the G action on X is a subset F of X with the property that, for each $x \in X$ the G-orbit $Gx = \{g(x) \mid g \in G\}$ intersects F at exactly one point.

A fundamental domain is by no means unique, but one cannot have proper inclusion of fundamental domains, i.e. if F_1 and F_2 are fundamental domains and $F_1 \subseteq F_2$ then $F_1 = F_2$. Moreover $\bigcup_{g \in G} g(F) = X$, where $g(F) = \{g(x) \mid x \in F\}$. Now consider the natural action of $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ on $\mathbb{Z}^n$ (also on $\mathbb{R}^n$). For each $\succcurlyeq \in \Omega$ define the following initial sets. (Note that $\succcurlyeq$ extends to $\mathbb{R}^n$ via Lemma 2.2.)

$$V_{\mathcal{G}}^{\succcurlyeq} := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \succcurlyeq \varphi\mathbf{v}, \text{ for all } \varphi \in \mathcal{G}\},$$
$$A_{\mathcal{G}}^{\succcurlyeq} := V_{\mathcal{G}}^{\succcurlyeq} \cap \mathbb{Z}^n = \{\mathbf{a} \in \mathbb{Z}^n \mid \mathbf{a} \succcurlyeq \varphi\mathbf{a}, \text{ for all } \varphi \in \mathcal{G}\}.$$

We will write $V^{\succcurlyeq}$ or $A^{\succcurlyeq}$ for simplicity without the subscript $\mathcal{G}$, if the group is understood from the context.

Lemma 3.1.

(i) $V^{\succcurlyeq}$ *and* $A^{\succcurlyeq}$ *are fundamental domains for the* $\mathcal{G}$*-action on* $\mathbb{R}^n$ *and* $\mathbb{Z}^n$ *respectively.*
(ii) $V^{\succcurlyeq}$ *is a convex cone of dimension* n *in* $\mathbb{R}^n$.
(iii) $\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}) = \Bbbk[\mathbf{x}^{A^{\succcurlyeq}}] = \Bbbk[\mathbf{x}^{\mathbf{a}}\colon \mathbf{a} \in A^{\succcurlyeq}]$.
(iv) *The map* $A^{\succcurlyeq} \mapsto \mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$ *is a bijection between* $\{A^{\succcurlyeq};\ \succcurlyeq \in \Omega\}$ *and* $\{\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}});\ \succcurlyeq \in \Omega\}$.

Proof. (i) By definition, for each $\mathbf{v} \in \mathbb{R}^n$ $\mathcal{G}\mathbf{v} \cap V^{\succcurlyeq}$ consists of the unique maximum of $\mathcal{G}\mathbf{v}$ under $\succcurlyeq$. Hence $V^{\succcurlyeq}$ is a fundamental domain for the $\mathcal{G}$-action on $\mathbb{R}^n$. Similarly for $A^{\succcurlyeq}$.

(ii) Let $\mathbf{v}_1, \mathbf{v}_2 \in V^{\succcurlyeq}$ and $r_1, r_2 \in \mathbb{R}_{\geqslant 0}$ then by the definition of $V^{\succcurlyeq}$ and Lemma 2.2 we have $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \succcurlyeq r_1\varphi(\mathbf{v}_1) + r_2\varphi(\mathbf{v}_2) = \varphi(r_1\mathbf{v}_1 + r_2\mathbf{v}_2)$ for all $\varphi \in \mathcal{G}$. Hence $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \in V^{\succcurlyeq}$, i.e. $V^{\succcurlyeq}$ is a convex cone. Next, since $V^{\succcurlyeq}$ is a fundamental domain then $\mathbb{R}^n = \bigcup_{\varphi \in \mathcal{G}} \varphi(V^{\succcurlyeq})$. Moreover $\dim(V^{\succcurlyeq}) = \dim \varphi(V^{\succcurlyeq})$ for all $\varphi \in \mathcal{G}$. Therefore, in view of Lemma 2.1, $\dim(V^{\succcurlyeq})$ cannot be smaller than n.

(iii) Note that $\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}) = \Bbbk[\mathbf{x}^{\mathrm{in}_{\succcurlyeq}(f)} \mid 0 \neq f \in \Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}]$. Hence we need to show that $\{\mathrm{in}_{\succcurlyeq}(f) \mid 0 \neq f \in \Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}\} = \mathsf{A}^{\succcurlyeq}$. But this fact follows easily from the observation that the orbit sums $\vartheta(\mathbf{x}^{\mathbf{a}}) := \sum_{\mathbf{b} \in \mathcal{G}\mathbf{a}} \mathbf{x}^{\mathbf{b}}$, $\mathbf{a} \in \mathbb{Z}^n$, form a $\Bbbk$-basis of the invariant algebra $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$ where $\mathcal{G}\mathbf{a}$ is the $\mathcal{G}$-orbit of $\mathbf{a}$. This last equality is also proved in [9, Lemma 2.6(a)].

(iv) This is an immediate consequence of (iii). □

Definition 3. The cone $V^{\succcurlyeq}$ is called the *initial convex cone* of $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$ associated to the monomial order $\succcurlyeq$.

3.2. Gröbner region

The concept of *Gröbner region* for ideals of polynomial rings was first introduced by Mora and Robbiano in [8], it is also studied by Sturmfels in [13]. We define its analogy for subalgebras as follows:

Definition 4 *(Gröbner region).* Let R be a subalgebra of $\Bbbk[\mathbf{x}]$ (or $\Bbbk[\mathbf{x}^{\pm 1}]$) and let R^* denote the non-zero elements of R. For each vector $\mathbf{w} \in \mathbb{R}^n$ the Gröbner region of R with respect to $\mathbf{w}$, denoted $\mathrm{GR}_{\mathbf{w}}(R)$, is the set of vectors $\mathbf{w}' \in \mathbb{R}^n$ such that

$$\bigl\{\mathrm{in}_{\succcurlyeq_{\mathbf{w}}}(f) \bigm| f \in R^*\bigr\} = \bigl\{\mathrm{in}_{\succcurlyeq_{\mathbf{w}'}}(f) \bigm| f \in R^*\bigr\}.$$

We like to throw in the zero vector for a reason that will be clear in the next lemma. In particular Gröbner region for the ring of multiplicative invariants is

$$\mathrm{GR}_{\mathbf{w}}\bigl(\Bbbk\bigl[\mathbf{x}^{\pm 1}\bigr]^{\mathcal{G}}\bigr) = \bigl\{\mathbf{w}' \in \mathbb{R}^n \bigm| A^{\succcurlyeq_{\mathbf{w}}} = A^{\succcurlyeq_{\mathbf{w}'}}\bigr\} \cup \{\mathbf{0}\}.$$

Lemma 3.2. $\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$ *is a convex cone for each* $\mathbf{w} \in \mathbb{R}^n$.

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$, i.e. $A^{\succcurlyeq_{\mathbf{v}_1}} = A^{\succcurlyeq_{\mathbf{v}_2}} = A^{\succcurlyeq_{\mathbf{w}}}$. Let $\mathbf{a} \in A^{\succcurlyeq_{\mathbf{w}}}$ then $\mathbf{a} \in A^{\succcurlyeq_{\mathbf{v}_1}} = A^{\succcurlyeq_{\mathbf{v}_2}}$, then for each $\varphi \in \mathcal{G}$, we have $\mathbf{a} \cdot \mathbf{v}_i \geqslant \varphi(\mathbf{a}) \cdot \mathbf{v}_i$, $i = 1, 2$. Now if at least one of the two inequalities is a strict inequality then we get $\mathbf{a} \cdot (\mathbf{v}_1 + \mathbf{v}_2) > \varphi(\mathbf{a}) \cdot (\mathbf{v}_1 + \mathbf{v}_2)$ which implies that $\mathbf{a} \succcurlyeq_{(\mathbf{v}_1+\mathbf{v}_2)} \varphi(\mathbf{a})$. On the other hand if $\mathbf{a} \cdot \mathbf{v}_i = \varphi(\mathbf{a}) \cdot \mathbf{v}_i,\ i = 1, 2$, then $\mathbf{a} \succcurlyeq_{\text{lex}} \varphi(\mathbf{a})$ and $\mathbf{a} \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \varphi(\mathbf{a}) \cdot (\mathbf{v}_1 + \mathbf{v}_2)$. By definition this means $\mathbf{a} \succcurlyeq_{(\mathbf{v}_1+\mathbf{v}_2)} \varphi(\mathbf{a})$. Therefore we have shown that $A^{\succcurlyeq_{\mathbf{w}}} \subseteq A^{\succcurlyeq_{(\mathbf{v}_1+\mathbf{v}_2)}}$. But since both are fundamental domains for the $\mathcal{G}$ action on $\mathbb{Z}^n$ (Lemma 3.1(i)) then the inclusion is an equality. Thus $A^{\succcurlyeq_{\mathbf{w}}} = A^{\succcurlyeq_{(\mathbf{v}_1+\mathbf{v}_2)}}$. As a result $\mathbf{v}_1 + \mathbf{v}_2 \in \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$. On the other hand it is clear from the definition that $\succcurlyeq_{\mathbf{u}} = \succcurlyeq_{\lambda\mathbf{u}}$ for any $\mathbf{u} \in \mathbb{R}^n$ and $\lambda > 0$. Hence if $\mathbf{u} \in \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$ then $\lambda\mathbf{u} \in \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$. Therefore $\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$ is a convex cone. $\square$

Example 1. Let

$$\mathcal{G}_1 = \left\{ \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_3(\mathbb{Z}).$$

Fix

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$$

and let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{Z}^3,$$

then

$$\begin{aligned} \mathbf{a} \in A^{\succcurlyeq_{\mathbf{w}}} \quad &\Leftrightarrow \quad \big(\mathbf{a} - \sigma(\mathbf{a})\big) \cdot \mathbf{w} > \mathbf{0} \vee \big[\big(\mathbf{a} - \sigma(\mathbf{a})\big) \cdot \mathbf{w} = \mathbf{0} \wedge \mathbf{a} \succcurlyeq_{\text{lex}} \sigma(\mathbf{a})\big] \\ &\Leftrightarrow \quad (2a_3 - a_2)w_3 > 0 \vee \big[(2a_3 - a_2)w_3 = 0 \wedge 2a_3 - a_2 \geqslant 0\big]. \end{aligned}$$

Now consider the following two possible cases on w_3:

Case (i) $w_3 \geqslant 0$. In this case

$$A^{\succcurlyeq_{\mathbf{w}}} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{Z}^3 \;\middle|\; 2a_3 - a_2 \geqslant 0 \right\}$$

and hence $A^{\succcurlyeq_{\mathbf{w}}} = A^{\succcurlyeq_{\mathbf{v}}}$ for some

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$$

iff $v_3 \geqslant 0$.

Case (ii) $w_3 < 0$. In this case

$$A^{\succcurlyeq_{\mathbf{w}}} = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{Z}^3 \;\middle|\; 2a_3 - a_2 \leqslant 0 \right\}.$$

In this case, $A^{\succcurlyeq_\mathbf{w}} = A^{\succcurlyeq_\mathbf{v}}$ iff $v_3 < 0$. Thus we have two Gröbner regions:

$$\mathrm{GR}_\mathbf{w}\big(\Bbbk\big[\mathbf{x}^{\pm 1}\big]^{\mathcal{G}_1}\big) = \begin{cases} \{\mathbf{v} = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right) \in \mathbb{R}^3 \mid v_3 \geqslant 0\} & \text{if } w_3 \geqslant 0, \\ \{\mathbf{v} = \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right) \in \mathbb{R}^3 \mid v_3 < 0\} & \text{if } w_3 < 0. \end{cases}$$

Example 2.

$$\mathcal{G}_2 = \left\{ \varphi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \varphi^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_3(\mathbb{Z}).$$

By definition

$$\big\{\mathbf{a} \in \mathbb{Z}^3 \bigm| \mathbf{a} \cdot \mathbf{w} > \varphi(\mathbf{a}) \cdot \mathbf{w}\big\} \subseteq A^{\succcurlyeq_\mathbf{w}} \subseteq \big\{\mathbf{a} \bigm| \mathbf{a} \cdot \mathbf{w} \geqslant \varphi(\mathbf{a}) \cdot \mathbf{w}\big\}.$$

But since $\varphi(\mathbf{a}) = -\mathbf{a}$ then above inclusion becomes

$$\big\{\mathbf{a} \in \mathbb{Z}^3 \bigm| \mathbf{a} \cdot \mathbf{w} > 0\big\} \subseteq A^{\succcurlyeq_\mathbf{w}} \subseteq \big\{\mathbf{a} \in \mathbb{R}^3 \bigm| \mathbf{a} \cdot \mathbf{w} \geqslant 0\big\}.$$

Now let $\mathbf{w}_1$ and $\mathbf{w}_2$ are non-zero and non-parallel vectors in $\mathbb{R}^3$ then it can easily be checked that $\{\mathbf{a} \in \mathbb{Z}^3 \mid \mathbf{a} \cdot \mathbf{w}_1 > 0 > \mathbf{a} \cdot \mathbf{w}_2\}$ is non-empty. It follows that $A^{\succcurlyeq_{\mathbf{w}_1}} \neq A^{\succcurlyeq_{\mathbf{w}_2}}$. Therefore $A^{\succcurlyeq_\mathbf{w}} = A^{\succcurlyeq_\mathbf{v}}$ iff $\mathbf{v} = \lambda \mathbf{w}$ for $\lambda > 0$ and hence, $\mathrm{GR}_\mathbf{w}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}) = \{\lambda \mathbf{w} \mid \lambda \in \mathbb{R}_{\geqslant 0}\}$. On the other hand if $\mathbf{w} = \mathbf{0}$ then $\succcurlyeq_\mathbf{0} = \succcurlyeq_{\mathbf{e}_1}$ where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore $A^{\succcurlyeq_\mathbf{0}} = A^{\succcurlyeq_{\mathbf{e}_1}}$ and hence $\mathrm{GR}_\mathbf{0}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}) = \mathrm{GR}_{\mathbf{e}_1}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}) = \{\lambda \mathbf{e}_1 \mid \lambda \in \mathbb{R}_{\geqslant 0}\}$.

3.3. Arrangement of reflecting hyperplanes

Let $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ and $\mathcal{R} := \{\sigma \in \mathcal{G} \mid \sigma \text{ is a reflection}\}$. If $\mathcal{R}$ is non-empty, then the set of hyperplanes $\{\mathcal{H}_\sigma := \ker(\mathrm{I} - \sigma),\ \sigma \in \mathcal{R}\}$ is called an *arrangement of reflecting hyperplanes for* $\mathcal{G}$. Let us define some terminologies following Bourbaki [1, Chap. V]. Each open connected component of $\mathbb{R}^n - \bigcup_{\sigma \in \mathcal{R}} \mathcal{H}_\sigma$ is called a *chamber* of the hyperplane arrangement. We shall denote the set of all such chambers by $\mathfrak{C}$. Note that $\mathfrak{C}$ is non-empty due to Lemma 2.1 and each chamber generates $\mathbb{R}^n$. A *wall* of a chamber C (or of its closure $\overline{C}$) is a hyperplane $\mathcal{H}_\sigma$ such that $\overline{C} \cap \mathcal{H}_\sigma$ is of co-dimension 1.

Definition 5 *($\mathcal{G}$-invariant inner product).* An inner product $\langle \_, \_ \rangle$ on $\mathbb{R}^n$ is called *$\mathcal{G}$-invariant* iff $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \varphi(\mathbf{u}), \varphi(\mathbf{v}) \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\varphi \in \mathcal{G}$. A $\mathcal{G}$-invariant inner product always exists, for example,

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{\varphi \in \mathcal{G}} \varphi(\mathbf{u}) \cdot \varphi(\mathbf{v}); \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Therefore from now on assume that $\langle \_, \_ \rangle$ is a $\mathcal{G}$-invariant inner product and endow $\mathbb{R}^n$ with the metric topology induced by this inner product. (Note that this topology is equivalent to the standard metric topology.) For each $\sigma \in \mathcal{R}$ the hyperplane $\mathcal{H}_\sigma = \ker(\mathrm{I} - \sigma)$ and the line $\ker(\mathrm{I} + \sigma)$ are orthogonal complements under $\langle \_, \_ \rangle$. Moreover let $\mathbf{w}_\sigma$ be a unit vector that generates $\ker(\mathrm{I} + \sigma)$, then σ will have the following explicit description.

$$\sigma(\mathbf{v}) = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{w}_\sigma \rangle \mathbf{w}_\sigma; \quad \mathbf{v} \in \mathbb{R}^n.$$

Now let $\succcurlyeq \in \Omega$, and $\mathbf{w}_\sigma$ as defined above. Replacing $\mathbf{w}_\sigma$ by $-\mathbf{w}_\sigma$ if necessary, let us also assume that $\mathbf{w}_\sigma \succcurlyeq \mathbf{0}$. Then,

$$\mathbf{v} \succcurlyeq \sigma(\mathbf{v}) \quad \Leftrightarrow \quad 2\langle \mathbf{v}, \mathbf{w}_\sigma \rangle \mathbf{w}_\sigma \succcurlyeq \mathbf{0} \quad \Leftrightarrow \quad \langle \mathbf{v}, \mathbf{w}_\sigma \rangle \geqslant 0. \tag{3.1}$$

Now let us denote the closed half space in $\mathbb{R}^n$ bounded by $\mathcal{H}_\sigma$, by

$$\mathcal{H}_\sigma^+ := \{ \mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{w}_\sigma \rangle \geqslant 0 \}.$$

Using Eq. (3.1) we have

$$\bigcap_{\sigma \in \mathcal{R}} \mathcal{H}_\sigma^+ = \bigcap_{\sigma \in \mathcal{R}} \{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \succcurlyeq \sigma(\mathbf{v}) \} \supseteq \bigcap_{\varphi \in \mathcal{G}} \{ \mathbf{v} \mid \mathbf{v} \succcurlyeq \varphi(\mathbf{v}) \} = V^{\succcurlyeq}. \tag{3.2}$$

Lemma 3.3. *Let $\mathcal{G}$ be a reflection group, $\mathcal{R}$ and $\mathfrak{C}$ as above. Then*

(i) *Each $\varphi \in \mathcal{G}$ permutes the reflecting hyperplanes $\{\mathcal{H}_\sigma \mid \sigma \in \mathcal{R}\}$.*
(ii) *$\mathcal{G}$ acts transitively on $\mathfrak{C}$ and $|\mathcal{G}| = |\mathfrak{C}|$.*
(iii) *For each $\succcurlyeq \in \Omega$ there exists $C \in \mathfrak{C}$ such that $V^{\succcurlyeq} = \overline{C}$.*

Proof. (i) Let $\varphi \in \mathcal{G}$ and $\sigma \in \mathcal{R}$ then $\varphi\sigma\varphi^{-1} \in \mathcal{R}$ and $\varphi(\mathcal{H}_\sigma) = \mathcal{H}_{\varphi\sigma\varphi^{-1}}$. Hence φ maps the set $\{\mathcal{H}_\sigma \mid \sigma \in \mathcal{R}\}$ to itself, with φ^{-1} its inverse map. Therefore φ permutes the reflecting hyperplanes.

(ii) First let us show $\mathcal{G}$ indeed acts on $\mathfrak{C}$. For this let $\varphi \in \mathcal{G}$ and $C \in \mathfrak{C}$. From (i) above $\varphi(C) \cap [\bigcup_{\sigma \in \mathcal{R}} \mathcal{H}_\sigma] = \emptyset$. Hence $\varphi(C) \subseteq C'$ for some $C' \in \mathfrak{C}$. But since $\varphi^{-1}(C') \cap C \neq \emptyset$ then $\varphi^{-1}(C') \subseteq C$. Hence $\varphi(C) = C'$. Now to show this action is transitive, let $C_1, C_2 \in \mathfrak{C}$. From Bourbaki [1, V.3.3, Theorem 2], $\overline{C_1}$ is a fundamental domain for the $\mathcal{G}$-action on $\mathbb{R}^n$. Hence there exists $\varphi \in \mathcal{G}$ such that $\varphi(\overline{C_1}) \cap C_2 \neq \emptyset$. Following the previous explanation $\varphi(C_1) \cap C_2 \neq \emptyset$ and hence $\varphi(C_1) = C_2$.

Finally to $|\mathcal{G}| = |\mathfrak{C}|$. Fix $C \in \mathfrak{C}$ and define a map

$$\mathcal{G} \to \mathfrak{C} : \varphi \mapsto \varphi(C).$$

This map is onto since $\mathcal{G}$ acts transitively on $\mathfrak{C}$. To show it is one-to-one, let $\varphi_1(C) = \varphi_2(C)$ then $\varphi_2^{-1} \circ \varphi_1(C) = C$. But again since $\overline{C}$ is a fundamental domain for the $\mathcal{G}$ action on $\mathbb{R}^n$ then $\varphi_2^{-1} \circ \varphi_1(\mathbf{c}) = \mathbf{c}$, $\forall \mathbf{c} \in C$. But since each chamber is n-dimensional, then C generates $\mathbb{R}^n$ and hence $\varphi_2^{-1} \circ \varphi_1 = \mathrm{I}$, equivalently $\varphi_1 = \varphi_2$.

(iii) Fix $\succcurlyeq \in \Omega$. Using the above settings in the definition of $\mathcal{H}_\sigma^+$ and Eq. (3.2), the intersection of the closed half spaces $\bigcap_{\sigma \in \mathcal{R}} \mathcal{H}_\sigma^+$ contains the n-dimensional cone $V^{\succcurlyeq}$. It follows that the intersection of open half spaces, $\bigcap_{\sigma\in\mathcal{R}}\{\mathbf{v}\in\mathbb{R}^n \mid \langle\mathbf{v},\mathbf{w}_\sigma\rangle > 0\}$ is non-empty and hence is a chamber, say C, for the collection of reflecting hyperplanes of $\mathcal{G}$. By [1, V.3.3, Theorem 2], its closure, $\overline{C}=\bigcap_{\sigma\in\mathcal{R}}\mathcal{H}_\sigma^+$, is a fundamental domain for the $\mathcal{G}$-action on $\mathbb{R}^n$. On the other hand by Lemma 3.1(i), $V^{\succcurlyeq}$ is also a fundamental domain for the $\mathcal{G}$-action on $\mathbb{R}^n$. Therefore the inclusion $V^{\succcurlyeq}\subseteq\bigcap_{\sigma\in\mathcal{R}}\mathcal{H}_\sigma^+=\overline{C}$ cannot be proper, hence $V^{\succcurlyeq}=\overline{C}$. $\square$

Remark 3. Let $\{\mathcal{H}_{\sigma_i}\}_{i=1}^t$ be all the walls of $\overline{C}=V^{\succcurlyeq}$. Applying [1, V.1.4, Proposition 9] we get $\overline{C}=\bigcap_{i=1}^t\mathcal{H}_{\sigma_i}^+$. Furthermore it can easily be shown from [1, V.3.4, Proposition 3(iii)] that the unit vectors $\mathbf{w}_{\sigma_i}\in\ker(\mathrm{I}+\mathcal{H}_{\sigma_i})$, $i=1,\dots,t$, are linearly independent. More details on this can also found in [15, Lemma 3.2].

3.4. *Further topological properties of $V^{\succcurlyeq}$*

Let $\partial(V^{\succcurlyeq})$, $\mathrm{int}(V^{\succcurlyeq})$ and $\overline{V^{\succcurlyeq}}$ denote the boundary, interior and closure of $V^{\succcurlyeq}$ respectively with the standard metric topology on $\mathbb{R}^n$.

Lemma 3.4. *For a finite group $\mathcal{G}\leqslant\mathrm{GL}_n(\mathbb{Z})$ as usual,*

- (i) $\mathrm{int}(V^{\succcurlyeq})\subseteq\bigcap_{\varphi\in\mathcal{G}-\{\mathrm{I}\}}\{\mathbf{v}\in\mathbb{R}^n \mid \mathbf{v}\succ\varphi(\mathbf{v})\}$.
- (ii) $\partial V^{\succcurlyeq_{\mathbf{w}}}\subseteq\bigcup_{\varphi\in\mathcal{G}-\{\mathrm{I}\}}\{\mathbf{v}\in\mathbb{R}^n \mid (\mathbf{v}-\varphi(\mathbf{v}))\cdot\mathbf{w}=0\}$.
- (iii) *If $\mathcal{G}$ is a non-reflection group then there exists $\mathbf{v}\in\partial V^{\succcurlyeq}$ such that $\mathbf{v}\neq\varphi(\mathbf{v})$ for all $\varphi\in\mathcal{G}-\{\mathrm{I}\}$.*

Proof. (i) Suppose there exists $\mathbf{v}\in\mathrm{int}(V^{\succcurlyeq})$ such that $\mathbf{v}=\varphi(\mathbf{v})$ for some $\varphi\in\mathcal{G}-\{\mathrm{I}\}$. Let $\mathbf{u}\in\mathbb{R}^n$ such that $\mathbf{u}\neq\varphi(\mathbf{u})$ (such $\mathbf{u}$ exists since $\varphi\neq\mathrm{I}$). Now since $V^{\succcurlyeq}$ is an n-dimensional cone and $\mathbf{v}\in\mathrm{int}(V^{\succcurlyeq})$ then $V^{\succcurlyeq}$ contains an open ball centered at $\mathbf{v}$. Therefore there exists $\lambda>0$ such that $\mathbf{v}\pm\lambda\mathbf{u}\in V^{\succcurlyeq}$. But since $\mathbf{v}\pm\lambda\mathbf{u}\neq\varphi(\mathbf{v}\pm\lambda\mathbf{u})$ then $\mathbf{v}\pm\lambda\mathbf{u}\succ\varphi(\mathbf{v}\pm\lambda\mathbf{u})=\varphi(\mathbf{v})\pm\lambda\varphi(\mathbf{u})=\mathbf{v}\pm\lambda\varphi(\mathbf{u})$. It follows that $\pm\mathbf{u}\succ\pm\varphi(\mathbf{u})$ a contradiction.

(ii) Let $\mathbf{v}\in\partial V^{\succcurlyeq_{\mathbf{w}}}$. If $(\mathbf{v}-\varphi(\mathbf{v}))\cdot\mathbf{w}\neq 0$ for some $\varphi\in\mathcal{G}-\{\mathrm{I}\}$, then by continuity of $\mathbf{w}\cdot(\mathrm{I}-\varphi)(\_\_):\mathbb{R}^n\to\mathbb{R}$, there exists a neighborhood U of $\mathbf{v}$ such that $\mathbf{w}\cdot(\mathbf{u}-\varphi(\mathbf{u}))\neq 0$ for all $\mathbf{u}\in U$. By assumption, $U\cap V^{\succcurlyeq_{\mathbf{w}}}\neq\emptyset$ and $U-V^{\succcurlyeq_{\mathbf{w}}}\neq\emptyset$. Then the map above takes positive values on restriction to $U\cap V^{\succcurlyeq_{\mathbf{w}}}$ and takes negative values on restriction to $U-V^{\succcurlyeq_{\mathbf{w}}}$. But as U is connected it will have a zero at some point in U contradicting the construction of U. Therefore $(\mathbf{v}-\varphi(\mathbf{v}))\cdot\mathbf{w}=0$ as required.

(iii) Assume to the contrary that

$$\partial V^{\succcurlyeq}\subseteq\bigcup_{\varphi\in\mathcal{G}-\{\mathrm{I}\}}\big\{\mathbf{v}\in\mathbb{R}^n \bigm| \mathbf{v}=\varphi(\mathbf{v})\big\}=\bigcup_{\varphi\in\mathcal{G}-\{\mathrm{I}\}}\ker(\mathrm{I}-\varphi). \tag{3.3}$$

Then it follows that $\overline{V^{\succcurlyeq}}$ is a polyhedral cone. (Note that, this fact is true in general independent of Eq. (3.3), since a finite union of these closed cones $\bigcup_{\varphi\in\mathcal{G}}\varphi(\overline{V^{\succcurlyeq}})=\mathbb{R}^n$ and any two of them intersect at most in their boundaries.) Now let $\mathcal{H}_1,\dots,\mathcal{H}_t$ denote all the walls of $\overline{V^{\succcurlyeq}}$ then from Eq. (3.3) and Lemma 2.1 $\mathcal{H}_i=\ker(\mathrm{I}-\sigma_i)$ for some $\sigma_i\in\mathcal{G}$. In this case σ_i is a reflection in $\mathcal{G}$. Let $\mathcal{H}_i^+$ denotes the closed half space bounded by $\mathcal{H}_i$ that contains $\overline{V^{\succcurlyeq}}$. Note that, the wall of a cone in Bourbaki's setting is the same as a *facet* in Fulton's notation [2] and hence by [2, p. 11, #(8)],

$\overline{V^{\succcurlyeq}} = \bigcap_{i=1}^{t} \mathcal{H}_i^+$. Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ generated by the wall reflections σ_i, $i = 1, \ldots, t$. Then by [1, V.3.3, Theorem 2], the polyhedral cone $\bigcap_{i=1}^{t} \mathcal{H}_i^+ = \overline{V^{\succcurlyeq}}$ is a fundamental domain for the $\mathcal{G}_0$ action on $\mathbb{R}^n$.

Claim. $\mathcal{G} = \mathcal{G}_0$.

By definition, $\mathcal{G}_0 \leqslant \mathcal{G}$. On the other hand, for any $\varphi \in \mathcal{G}$, there exists $\varphi_0 \in \mathcal{G}_0$ such that $\varphi(\overline{V^{\succcurlyeq}}) \cap \varphi_0(\overline{V^{\succcurlyeq}})$ has non-empty interior, say $\varphi(\mathbf{v}_1) = \varphi_0(\mathbf{v}_2)$ for some $\mathbf{v}_1, \mathbf{v}_2 \in \operatorname{int}(V^{\succcurlyeq})$. Equivalently $\varphi^{-1}\varphi_0(\mathbf{v}_2) = \mathbf{v}_1$. But since $V^{\succcurlyeq}$ is a fundamental domain for the $\mathcal{G}$-action on $\mathbb{R}^n$ then $\mathbf{v}_2 = \mathbf{v}_1$ and hence $\varphi^{-1}\varphi_0(\mathbf{v}_2) = \mathbf{v}_2$. By (i) above $\varphi^{-1}\varphi_0 = \mathrm{I}$ which implies that $\varphi = \varphi_0 \in \mathcal{G}_0$. Therefore $\mathcal{G} = \mathcal{G}_0$ as claimed. This contradicts the hypothesis, proving (iii). $\square$

The inclusions in Lemma 3.4[(i) and (ii)] above may not necessarily be equalities as the following example demonstrates.

Example 3. Let $\mathcal{G} = \langle\varphi\rangle$ where φ is the counterclockwise rotation by $\pi/2$ on $\mathbb{R}^2$. $\mathcal{G}$ is a cyclic group of order 4. Its elements are:

$$\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varphi^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \varphi^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varphi^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fix $\mathbf{w} = \binom{2}{1} \in \mathbb{R}^2$. Then

$$\mathbf{v} \in V^{\succcurlyeq_{\mathbf{w}}} \quad \Leftrightarrow \quad \left[\mathbf{v}\cdot\mathbf{w} > \varphi^i(\mathbf{v})\cdot\mathbf{w}\right] \vee \left[\left(\mathbf{v}\cdot\mathbf{w} = \varphi^i(\mathbf{v})\cdot\mathbf{w}\right) \wedge \left(\mathbf{v} \succcurlyeq_{\mathrm{lex}} \varphi^i(\mathbf{v})\right)\right]$$

for each $i = 1, 2, 3$. Solving the three inequalities simultaneously we get that

$$\begin{aligned} V^{\succcurlyeq_{\mathbf{w}}} &= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; v_1 + 3v_2 \geqslant 0 \wedge 2v_1 + v_2 > 0 \wedge 3v_1 - v_2 > 0 \right\} \\ &= \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; v_1 + 3v_2 \geqslant 0 \wedge 3v_1 - v_2 > 0 \right\}. \end{aligned}$$

It follows that

$$\partial V^{\succcurlyeq_{\mathbf{w}}} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; v_1 \geqslant 0 \wedge \left[(v_1 + 3v_2 = 0) \vee (3v_1 - v_2 = 0)\right] \right\}.$$

- To show Lemma 3.4(i) is proper inclusion, pick $\mathbf{a} = \binom{3}{-1} \in \partial V^{\succcurlyeq_{\mathbf{w}}}$. Then

$$\mathbf{a} \succ_{\mathbf{w}} \varphi(\mathbf{a}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \mathbf{a} \succ_{\mathbf{w}} \varphi^2(\mathbf{a}) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \mathbf{a} \succ_{\mathbf{w}} \varphi^3(\mathbf{a}) = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

  Therefore $\mathbf{a} \in \bigcap_{\varphi\in\mathcal{G}-\{\mathrm{I}\}} \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \succ_{\mathbf{w}} \varphi(\mathbf{v})\}$ but $\mathbf{a} \notin \operatorname{int}(V^{\succcurlyeq_{\mathbf{w}}})$.
- To show Lemma 3.4(ii) is proper inclusion, observe that the vector $\mathbf{b} = \binom{1}{-2}$ is not an element of $\partial V^{\succcurlyeq_{\mathbf{w}}}$. But $[\mathbf{b} - \varphi^2(\mathbf{b})]\cdot\mathbf{w} = \binom{2}{-4}\cdot\binom{2}{1} = 0$.

Theorem 3.5. *Let $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ be a non-reflection group and let $\succcurlyeq_{\mathbf{w}}$ be the monomial order induced by $\mathbf{w} \in \mathbb{R}^n$. Then there exists a sequence $\mathbf{w}_n$ converging to $\mathbf{w}$ such that $A^{\succcurlyeq_{\mathbf{w}_n}} \neq A^{\succcurlyeq_{\mathbf{w}}}$.*

Proof. Let $\mathbf{v} \in \partial V^{\succcurlyeq_{\mathbf{w}}}$ such that $\mathbf{v} \neq \varphi(\mathbf{v})$ for all $\varphi \in \mathcal{G} - \{\mathrm{I}\}$ (such a vector exists by Lemma 3.4(iii)). Now using Lemma 3.4(ii) let us choose $\varphi_0 \in \mathcal{G} - \{\mathrm{I}\}$ such that $(\varphi_0(\mathbf{v}) - \mathbf{v}) \cdot \mathbf{w} = 0$. Define $\mathbf{w}_n := \mathbf{w} + \frac{1}{n}(\varphi_0(\mathbf{v}) - \mathbf{v})$ and let f_n be a sequence of functions given by

$$f_n : \mathbb{R}^n \to \mathbb{R} : \mathbf{u} \mapsto \mathbf{w}_n \cdot (\varphi_0 - \mathrm{I})(\mathbf{u}).$$

Each f_n is continuous. Moreover for the above choice of vector $\mathbf{v} \in \partial V^{\succcurlyeq_{\mathbf{w}}}$ we have

$$\begin{aligned} f_n(\mathbf{v}) &= \left[\mathbf{w} + \frac{1}{n}\big(\varphi_0(\mathbf{v}) - \mathbf{v}\big)\right] \cdot \big(\varphi_0(\mathbf{v}) - \mathbf{v}\big) \\ &= \frac{1}{n}\big\|\varphi_0(\mathbf{v}) - \mathbf{v}\big\|^2 > 0. \end{aligned}$$

Hence there exists $\epsilon_n > 0$ such that $f_n > 0$ on $B_{\epsilon_n}(\mathbf{v}) = \{\mathbf{u} \in \mathbb{R}^n \colon \|\mathbf{u} - \mathbf{v}\| < \epsilon_n\}$. Now since the rational points in $\mathbb{R}^n$ are dense then there exists $\mathbf{v}_n \in \mathbb{Q}^n$ such that $\mathbf{v}_n \in \mathrm{int}(V^{\succcurlyeq_{\mathbf{w}}}) \cap B_{\epsilon_n}(\mathbf{v})$. Let λ_n be a positive common denominator of the coordinates of $\mathbf{v}_n$, then $\lambda_n \mathbf{v}_n \in \mathbb{Z}^n \cap V^{\succcurlyeq_{\mathbf{w}}} = A^{\succcurlyeq_{\mathbf{w}}}$. But on the other hand,

$$\mathbf{w}_n \cdot \varphi_0(\lambda_n \mathbf{v}_n) - \mathbf{w}_n \cdot (\lambda_n \mathbf{v}_n) = \lambda_n \mathbf{w}_n \cdot \big(\varphi_0(\mathbf{v}_n) - \mathbf{v}_n\big) = \lambda_n f_n(\mathbf{v}_n) > 0.$$

It follows that $\varphi_0(\lambda_n \mathbf{v}_n) \succ_{\mathbf{w}_n} \lambda_n \mathbf{v}_n$ and hence $\lambda_n \mathbf{v}_n \notin A^{\succcurlyeq_{\mathbf{w}_n}}$. Therefore $A^{\succcurlyeq_{\mathbf{w}_n}} \neq A^{\succcurlyeq_{\mathbf{w}}}$. Now the result follows since $\mathbf{w}_n = \mathbf{w} + \frac{1}{n}(\varphi_0(\mathbf{v}) - \mathbf{v}) \xrightarrow{n \to \infty} \mathbf{w}$. $\square$

Theorem 3.5 is a generalization of [6, Lemma 2.5].

Corollary 3.6. *If $\mathcal{G}$ is a non-reflection group then* $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}))$ *is at most $n - 1$ for all $\mathbf{w} \in \mathbb{R}^n$.*

Proof. If $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = n$ then since $\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$ is a convex cone in $\mathbb{R}^n$ (Lemma 3.2), it will contain an open ball in $\mathbb{R}^n$, but this contradicts Theorem 3.5. Therefore $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) \leqslant n - 1$. $\square$

4. Main results

In this section we will state and prove the main results that are stated briefly in the introduction.

4.1. Distinct initial algebras of $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}$

The main result of this section is to calculate the cardinality of the set of distinct initial algebras of the ring of multiplicative invariants. But first let us start with some examples.

Example 4. Consider the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ given in Examples 1 and 2 of Section 3.2. Both groups are subgroups of $\mathrm{GL}_3(\mathbb{Z})$ of order 2. The difference between them is that $\mathcal{G}_1$ is a reflection group but $\mathcal{G}_2$ is not. Now let us calculate the number of distinct initial algebras for $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_1}$ and $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}$.

(1) *Distinct initial algebras of* $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_1}$: Recall that the reflection σ generates the group $\mathcal{G}_1$. As usual let $\mathbf{w}_\sigma$ denote a unit vector in $\ker(\mathrm{I}+\sigma)$ w.r.t. a $\mathcal{G}$-invariant inner product. Now for any $\succcurlyeq \in \Omega$, we have the following two cases on $\mathbf{w}_\sigma$

Case i. $\mathbf{w}_\sigma \succ \mathbf{0}$. Applying Eq. (3.1) we have

$$A^{\succcurlyeq} = \{\mathbf{a} \in \mathbb{Z}^3 \mid \mathbf{a} \succcurlyeq \sigma(\mathbf{a})\} = \{\mathbf{a} \in \mathbb{Z}^3 \mid \langle \mathbf{a}, \mathbf{w}_\sigma \rangle \geqslant 0\}.$$

Case ii. $\mathbf{0} \succ \mathbf{w}_\sigma$. In this case $-\mathbf{w}_\sigma \succ \mathbf{0}$ and hence

$$A^{\succcurlyeq} = \{\mathbf{a} \in \mathbb{Z}^3 \mid \langle \mathbf{a}, -\mathbf{w}_\sigma \rangle \geqslant 0\} = \{\mathbf{a} \in \mathbb{Z}^3 \mid \langle \mathbf{a}, \mathbf{w}_\sigma \rangle \leqslant 0\}.$$

Therefore there are exactly two distinct $A^{\succcurlyeq}$, and hence

$$\left|\left\{\mathrm{in}_{\succcurlyeq}\left(\Bbbk\left[\mathbf{x}^{\pm 1}\right]^{\mathcal{G}_1}\right): \succcurlyeq \in \Omega\right\}\right| = 2 = |\mathcal{G}_1|.$$

(2) *Distinct initial algebras of* $\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}$: For each $\mathbf{w} \in \mathbb{R}^3 - \{\mathbf{0}\}$, we showed in Example 2 that $\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2}) = \{\lambda \mathbf{w}\colon \lambda \in \mathbb{R}_{\geqslant 0}\}$. But since there are uncountably many pairwise distinct lines in $\mathbb{R}^3$ that pass through the origin, (Lemma 2.1), then $|\{\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_2})\colon \mathbf{w} \in \mathbb{R}^3\}| = c$. Therefore, since $|\Omega| = c$, we have

$$\left|\left\{\mathrm{in}_{\succcurlyeq}\left(\Bbbk\left[\mathbf{x}^{\pm 1}\right]^{\mathcal{G}_2}\right): \succcurlyeq \in \Omega\right\}\right| = c.$$

The above examples generalize to the following fact on cardinality of distinct initial algebras.

Theorem 4.1. *Let $\mathcal{G}$ be a (finite) subgroup of $\mathrm{GL}_n(\mathbb{Z})$, then the cardinality of the set $\{\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})\colon \succcurlyeq \in \Omega\}$ of distinct initial algebras of the ring of multiplicative invariants is exactly $|\mathcal{G}|$ iff $\mathcal{G}$ is a reflection group and the continuum c if $\mathcal{G}$ is not a reflection group.*

Proof. Case (i): *$\mathcal{G}$ is a reflection group*. Let $\succcurlyeq \in \Omega$ then by Lemma 3.3(iii) there exists $C \in \mathfrak{C}$ such that $V^{\succcurlyeq} = \overline{C}$. This fact together with Lemma 3.3(ii) gives us

$$\left|\left\{A^{\succcurlyeq}\colon \succcurlyeq \in \Omega\right\}\right| = \left|\left\{V^{\succcurlyeq}\colon \succcurlyeq \in \Omega\right\}\right| \leqslant |\mathfrak{C}| = |\mathcal{G}|. \tag{4.1}$$

On the other hand let C' be an arbitrary chamber in $\mathfrak{C}$. Fix $\succcurlyeq \in \Omega$ and let $V^{\succcurlyeq} = \overline{C}$ for some $C \in \mathfrak{C}$. Now since $\mathcal{G}$ acts transitively on $\mathfrak{C}$, there exists $\varphi_0 \in \mathcal{G}$ such that $\varphi_0(C') = C$. Now define an ordering $\succcurlyeq'$ on $\mathbb{Z}^n$ by $\mathbf{a} \succcurlyeq' \mathbf{b}$ iff $\varphi_0\mathbf{a} \succcurlyeq \varphi_0\mathbf{b}$. It can be verified that $\succcurlyeq' \in \Omega$. Hence for each $\mathbf{v} \in C'$ we have $\varphi_0\mathbf{v} \succcurlyeq \varphi(\varphi_0\mathbf{v}) = \varphi_0(\varphi_0^{-1}\varphi\varphi_0)\mathbf{v}$: $\forall \varphi \in \mathcal{G}$. It follows by definition of $\succcurlyeq'$ that $\mathbf{v} \succcurlyeq' (\varphi_0^{-1}\varphi\varphi_0)\mathbf{v}$ for all $\varphi \in \mathcal{G}$. But since $\{\varphi_0\varphi\varphi_0^{-1}\colon \varphi \in \mathcal{G}\} = \mathcal{G}$ then $\mathbf{v} \in V^{\succcurlyeq'}$. Therefore $C' \subseteq V^{\succcurlyeq'}$ and hence $\overline{C'} = V^{\succcurlyeq'}$. It follows that $|\mathfrak{C}| \leqslant |\{V^{\succcurlyeq}\colon \succcurlyeq \in \Omega\}|$. Combining this with Eq. (4.1) we get that $|\{A^{\succcurlyeq}\colon \succcurlyeq \in \Omega\}| = |\mathcal{G}|$. Therefore by Lemma 3.1(iv), the cardinality of the set of distinct initial algebras of the ring of invariants is $|\mathcal{G}|$.

Case (ii): *$\mathcal{G}$ is a non-reflection group*. From the construction of Gröbner regions and initial algebras we have

$$\left|\left\{\mathrm{GR}_{\mathbf{w}}\left(\Bbbk\left[\mathbf{x}^{\pm 1}\right]^{\mathcal{G}}\right);\ \mathbf{w} \in \mathbb{R}^n\right\}\right| \leqslant \left|\left\{\mathrm{in}_{\succcurlyeq}\left(\Bbbk\left[\mathbf{x}^{\pm 1}\right]^{\mathcal{G}}\right);\ \succcurlyeq \in \Omega\right\}\right| \leqslant |\Omega| = c.$$

Applying Corollary 3.6 we have, $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) \leqslant n-1$ for each $\mathbf{w} \in \mathbb{R}^n$. But since $\bigcup_{\mathbf{w}\in\mathbb{R}^n} \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}}) = \mathbb{R}^n$, then by Lemma 2.1 we cannot cover $\mathbb{R}^n$ by a countable collection of the subspaces spanned by each Gröbner region. Hence $|\{\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}});\ \mathbf{w} \in \mathbb{R}^n\}| = c$. Therefore $|\{\mathrm{in}_{\succcurlyeq}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}});\ \succcurlyeq \in \Omega\}| = c$, completing the proof. □

4.2. *Dimension of Gröbner region*

In this section we will give a characterization of the ring of multiplicative invariants in terms of the dimension of Gröbner regions.

First, for each $\mathsf{k} \in \{1,\dots,n\}$ we construct a subgroup $\mathcal{G}_{\mathsf{k}}$ of $\mathrm{GL}_n(\mathbb{Z})$ where $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = \mathsf{k}$. Denote by I_s the identity matrix of rank $s \in \{1,\dots,n\}$. We call $\sigma \in \mathrm{GL}_n(\mathbb{Z})$ a k-*reflection* if the rank of $(\sigma - \mathrm{I}_n)$ is at most k. A 1-reflection is the standard hyperplane reflection defined earlier.

Example 5. Let $\mathcal{G}_{\mathsf{k}}$ be the cyclic group generated by the $(n+1-\mathsf{k})$-reflection

$$\sigma_{n+1-\mathsf{k}} := \begin{pmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} -\mathrm{I}_{(n+1-\mathsf{k})} & 0 \\ 0 & \mathrm{I}_{(\mathsf{k}-1)} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}).$$

Consider the usual projection

$$\pi : \mathbb{R}^n \to \mathbb{R}^{n+1-\mathsf{k}}$$

where $\pi(\mathbf{v})$ is the vector consisting of the first $n+1-\mathsf{k}$ coordinates of $\mathbf{v} \in \mathbb{R}^n$. Now for each $\mathbf{w} \in \mathbb{R}^n - \{\mathbf{0}\}$, by definition of $A^{\succcurlyeq_{\mathbf{w}}}$ we have

$$\{\mathbf{a} \in \mathbb{Z}^n \mid \pi(\mathbf{w}) \cdot \pi(\mathbf{a}) > 0\} \subseteq A^{\succcurlyeq_{\mathbf{w}}} \subseteq \{\mathbf{a} \in \mathbb{Z}^n \mid \pi(\mathbf{w}) \cdot \pi(\mathbf{a}) \geqslant 0\}.$$

Therefore if $\pi(\mathbf{w}) = \lambda\pi(\mathbf{u})$ for some $\lambda > 0$ then $A^{\succcurlyeq_{\mathbf{w}}} = A^{\succcurlyeq_{\mathbf{u}}}$. On the other hand if $\pi(\mathbf{w})$ and $\pi(\mathbf{u})$ are not parallel it is not hard to see that

$$\{\mathbf{a} \in \mathbb{Z}^n \mid \pi(\mathbf{w}) \cdot \pi(\mathbf{a}) > \mathbf{0} > \pi(\mathbf{u}) \cdot \pi(\mathbf{a})\}$$

is a non-empty subset of $\mathbb{Z}^n$. Clearly elements of the above set belong to $A^{\succcurlyeq_{\mathbf{w}}}$ but not to $A^{\succcurlyeq_{\mathbf{u}}}$, and hence $A^{\succcurlyeq_{\mathbf{w}}} \neq A^{\succcurlyeq_{\mathbf{u}}}$. Therefore,

$$\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_{\mathsf{k}}}) = \{\mathbf{u} \in \mathbb{R}^n \mid \pi(\mathbf{u}) = \lambda\pi(\mathbf{w})\}.$$

It follows that $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}_{\mathsf{k}}})) = \dim(\ker(\pi)) + 1 = \mathsf{k}$.

Observe that $\sigma_{n+1-\mathsf{k}}$ is a reflection (1-reflection) if $\mathsf{k} = n$. In that case, the dimension of the Gröbner region is n. This fact is true in general for all reflection groups as shown below.

Theorem 4.2. *Let $\mathcal{G} \leqslant \mathrm{GL}_n(\mathbb{Z})$ then $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = n$ for all $\mathbf{w} \in \mathbb{R}^n$ if and only if $\mathcal{G}$ is a reflection group.*

Proof. If $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = n$ then by Corollary 3.6, $\mathcal{G}$ must be a reflection group. Conversely assume $\mathcal{G}$ is a reflection group. Then in view of Theorem 4.1 there are only finitely many $(= |\mathcal{G}|)$ distinct Gröbner regions of the invariant algebra and hence $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = n$ for some $\mathbf{w} \in \mathbb{R}^n$. But it is not clear if the dimension of the Gröbner region is n for all $\mathbf{w} \in \mathbb{R}^n$. (Note that unlike the initial convex cones, Gröbner regions cannot be permuted by $\mathcal{G}$.) Now let $\mathbf{w} \in \mathbb{R}^n$ be arbitrary and let $V^{\succcurlyeq_{\mathbf{w}}}$ be the initial convex cone w.r.t. the $\mathcal{G}$-action. Using Remark 3, let $\mathbf{w}_{\sigma_1}, \ldots, \mathbf{w}_{\sigma_t}$ be the linearly independent vectors such that $\mathbf{w}_{\sigma_i}^{\perp} = \mathcal{H}_{\sigma_i}$ consists of all the walls of $V^{\succcurlyeq_{\mathbf{w}}}$. Then $V^{\succcurlyeq_{\mathbf{w}}} = \bigcap_{i=1}^{t} \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{w}_{\sigma_i} \rangle \geqslant 0\}$. Consider now the set $\mathfrak{O}_{\mathbf{w}} := \bigcap_{i=1}^{t} \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w}_{\sigma_i} > 0\}$. Clearly $\mathfrak{O}_{\mathbf{w}}$ is open in $\mathbb{R}^n$. Moreover using the fact that the vectors $\mathbf{w}_i$ are linearly independent, it is not hard to show that $\mathfrak{O}_{\mathbf{w}}$ is non-empty. See for example [15, Theorem 4.2] for details. Now for any $\mathbf{v} \in \mathfrak{O}_{\mathbf{w}}$ we have

$$
\begin{aligned}
A^{\succcurlyeq_{\mathbf{v}}} \subseteq \bigcap_{i=1}^{t} \bigl\{\mathbf{a} \in \mathbb{Z}^n \bigm| \mathbf{a} \succcurlyeq_{\mathbf{v}} \sigma_i(\mathbf{a})\bigr\} &= \bigcap_{i=1}^{t} \bigl\{\mathbf{a} \bigm| \langle \mathbf{a}, \mathbf{w}_{\sigma_i} \rangle \mathbf{w}_{\sigma_i} \cdot \mathbf{v} \geqslant 0\bigr\} \\
&= \bigcap_{i=1}^{t} \bigl\{\mathbf{a} \bigm| \langle \mathbf{a}, \mathbf{w}_{\sigma_i} \rangle \geqslant 0\bigr\} = A^{\succcurlyeq_{\mathbf{w}}}.
\end{aligned}
$$

On the other hand, since both are fundamental domains for the $\mathcal{G}$ action on $\mathbb{Z}^n$ then the above inclusion must be equality, i.e. $A^{\succcurlyeq_{\mathbf{v}}} = A^{\succcurlyeq_{\mathbf{w}}}$. Hence $\mathfrak{O}_{\mathbf{w}} \subseteq \mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})$. But since $\mathfrak{O}_{\mathbf{w}}$ is a non-empty open subset of $\mathbb{R}^n$ then $\dim(\mathrm{GR}_{\mathbf{w}}(\Bbbk[\mathbf{x}^{\pm 1}]^{\mathcal{G}})) = n$. This completes the proof. $\square$

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References

[1] N. Bourbaki, Groupes et Algèbres de Lie, chap. 4–6, Hermann, Paris, 1968.

[2] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Stud., vol. 131, Princeton University Press, Princeton, NJ, 1993.

[3] M. Göbel, A constructive description of SAGBI bases for polynomial invariants of permutation groups, J. Symbolic Comput. 26 (1998) 261–272.

[4] M. Göbel, The optimal lower bound for generators of invariant rings without finite SAGBI basis with respect to any admissible order, Discrete Math. Theor. Comput. Sci. 3 (2) (1999) 65–70 (electronic).

[5] D. Kapur, K. Madlener, A completion procedure for computing a canonical bases for a k-subalgebra, in: E. Kaltofen, S. Watt (Eds.), Proceedings of Computers and Mathematics, '89, MIT, Cambridge, MA, 1989, pp. 1–11.

[6] S. Kuroda, The infiniteness of the SAGBI bases for certain invariant rings, Osaka J. Math. 39 (2002) 665–680.

[7] M. Lorenz, Multiplicative invariant theory, in: Invariant Theory and Algebraic Transformation Groups, in: Encyclopaedia Math. Sci., vol. 135, Springer-Verlag, Berlin/Heidelberg, 2005.

[8] T. Mora, L. Robbiano, The Gröbner fan of an ideal, J. Symbolic Comput. 6 (1988) 183–208.

[9] Z. Reichstein, SAGBI bases in rings of multiplicative invariants, Comment. Math. Helv. 78 (1) (2003) 185–202.

[10] L. Robbiano, Term orders on the polynomial ring, in: Proceedings of the EUROCAL '85, in: Lect. Notes Comput. Sci., vol. 204, Springer-Verlag, 1985, pp. 513–517.

[11] L. Robbiano, On the theory of graded structures, J. Symbolic Comput. 2 (1986) 139–170.

[12] L. Robbiano, M. Sweedler, Subalgebra bases, in: Commutative Algebra, Salvador, 1988, in: Lecture Notes in Math., vol. 1430, Springer-Verlag, Berlin, 1990, pp. 61–87.
[13] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Ser., vol. 8, Amer. Math. Soc., 1995.
[14] M. Tesemma, On multiplicative invariants of finite reflection groups, Comm. Algebra 35 (2007) 2258–2274.
[15] M. Tesemma, H. Wang, Archimedean orders on certain rings of invariants, Algebra Colloq., in press.
[16] N.M. Thiéry, S. Thomassé, Convex cones and SAGBI bases of permutation invariants, in: CRM Proc. Lecture Notes, vol. 35, 2004, pp. 259–263.